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Damped dynamic response determination in the frequency domain for partially damped large scale structures

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ABSTRACT

A methodology is developed to solve the modal frequency response problem for the structural system with partially distributed structural and viscous damping materials and/or components. For the problems of interest, it is noted that the finite element viscous and structural damping matrices are typically very sparse, so the rank of the matrices are identified with the singular value decomposition (SVD) method. Then the modal frequency response problem is reformulated with the low rank matrices obtained from the SVD method. The strategy of the new approach is to compute the modal solution using the Sherman–Morrison–Woodbury formula for the inverse of equation which is subjected to low rank modifications, instead of factoring the coefficient matrix at each excitation frequency. Numerical results are presented to validate and assess the proposed approach, and the advantages of this method are examined.

1. Introduction

As part of a structural evaluation, a steady-state response of the structural system has been carried out in the frequency domain. Today, the determination of frequency response behavior for structures is performed with the finite element (FE) method, but the frequency response analysis in terms of all of the finite element degrees of freedom in the millions has been prohibitive for large scale structures. Instead, industry has performed the frequency response analysis using the modal analysis [1].

For a system without damping or with proportional damping, it is trivial to solve the modal frequency response problem because the coefficient matrix becomes uncoupled [2]. However, because the dynamic response of structural system is susceptible to damping effects [3], it is essential to describe realistic damping distribution.

The non-proportional damping can describe the realistic damping of structures. The non-proportional damping results in a coupled problem in the modal formulation.

With most structures, a relatively small amount of non-proportional damping provides a large reduction in stress and deflection by dissipating energy from the structure. Viscous and structural damping are the most commonly used types of non-proportional damping [3]. Generally, the energy loss of elastic materials is described by the structural damping. One of the most common types of the viscous damping is a piston that is attached to the structural body and is arranged to move through liquid or air in a cylinder or bellows. When these non-proportional dampings are considered in the finite element model, the corresponding coupled modal frequency response problem has been solved with either direct methods or

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iterative methods. These approaches, however, are too expensive for large scale FE models that require more than thousands of modes to represent the dynamic responses [1,4].

Recently, an efficient algorithm, fast frequency response analysis (FFRA) algorithm, is developed for modal frequency response analysis considering the structural damping [1] and viscous damping [5], in which the FFRA algorithm dramatically improved the performance of the modal frequency response analysis compared to conventional approaches [2]. In this approach, the complex symmetric matrix eigenvalue problem is required regardless of the way of damping distribution. However, the complex symmetric matrix eigenvalue problem still requires more reliable mathematical theory due to quasi-null condition and an expensive computation cost [6,7].

This paper presents a new algorithm as an extension of the FFRA algorithm for the dynamic response analysis in the frequency domain to consider the partially damped structural system with viscous and structural damping, which is a realistic situation in structural systems. The advantage of this new approach is that it does not require the complex symmetric eigenvalue problem any more, which is the most expensive part in the FFRA algorithm [1]. The accuracy and efficiency are demonstrated by a real industry vehicle finite element model.

2. Problem formulation

The forced vibration equations of motion for damped structures discretized by the finite element method are

$$[M]\ddot{\mathbf{x}}(t)+[B]\dot{\mathbf{x}}(t)+(1+\mathrm{i}\gamma)[K]\mathbf{x}(t)+\mathrm{i}[K_S]\mathbf{x}(t)=\mathbf{p}(t) \tag{1}$$

where the scalar γ is a global structural damping coefficient and $\mathrm{i}=\sqrt{-1}$. $[M]$, $[B]$, $[K]$, and $[K_S]\in\mathbb{R}^{n\times n}$ are the finite element mass, non-proportional viscous damping, stiffness matrix, and local structural damping matrix, respectively. The finite element local structural damping matrix $[K_S]$ represents localized deviations of specific elements from the global structural damping γ.

In a structural system, a damping treatment consists of any material, combination of materials, or components in order to increase its ability to dissipate mechanical energy. The focus of this paper is to consider a structural system that consists of only a few damping materials or components. Fig. 1 illustrates a structural system finite element model, which has a large number of finite element degrees of freedom, but only a few degrees of freedom for the structural and viscous damping finite elements. In this case, it is noted that the finite element viscous and structural damping matrices are typically very sparse for problems of interest in structural systems, which results in the low rank of finite element viscous and structural damping matrices. For example, in full or large scale vehicle finite element model, the small number of viscous damping finite element elements, which describes engine mounts and shock absorbers, and structural damping finite element elements, which describes floor and suspension spring system, are used.

In order to obtain the frequency response formulation for the equations of motion (1), we assume a harmonic solution of the form $\mathbf{x}(t)=\mathbf{X}(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\in\mathbb{C}^{n\times nf}$ for a harmonic excitation $\mathbf{p}(t)=\mathbf{P}(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\in\mathbb{C}^{n\times nf}$, in which ω is the radian frequency of time-harmonic excitation and nf is the number of load cases. When the first and second derivatives of $\mathbf{x}(t)$ are substituted into Eq. (1), the following is obtained after dividing by $\mathrm{e}^{\mathrm{i}\omega t}$:

$$\{-\omega^2[M]+\mathrm{i}\omega[B]+(1+\mathrm{i}\gamma)[K]+\mathrm{i}[K_S]\}\mathbf{X}(\omega)=\mathbf{P}(\omega) \tag{2}$$

This is a system of equations for the direct frequency response analysis. For excitations $\mathbf{P}(\omega)\in\mathbb{C}^{n\times nf}$, the frequency responses $\mathbf{X}(\omega)\in\mathbb{C}^{n\times nf}$ are calculated at each excitation frequency ω by solving a set of complex linear equations (2).

Recently, the size n of FE models increases to over millions of degrees of freedom as the accuracy requirement of analysis increases. However, solving these very large FE systems of equations at many frequencies has been prohibitive because CPU time, memory, and data I/O are limited, even though this method is very straightforward. Instead, modal frequency response analysis has been used [1,5].

To formulate the modal frequency response problem, the frequency response problem (2) is projected onto the space spanned by eigenvectors in $[\Phi]\in\mathbb{R}^{n\times m}$ of a partial eigensolution of the generalized eigenvalue problem $[K][\Phi]=[M][\Phi][\Lambda]$, in which $[\Lambda]\in\mathbb{R}^{m\times m}$ is a eigenvalue matrix and m is the number of modes obtained up to cutoff frequency ($m\ll n$) [1]. By

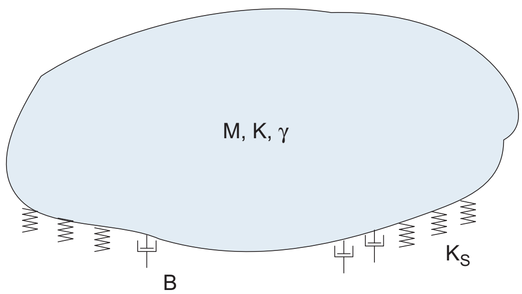

Fig. 1. A large scale finite element model with only a few viscous and structural damping elements.

substituting $\mathbf{X}(\omega) = [\Phi]\mathbf{Z}(\omega)$ and premultiplying by $[\Phi]^{\mathrm{T}}$, the modal frequency response problem is represented as

$$\{-\omega^2[I] + \mathrm{i}\omega[\bar{B}] + (1+\mathrm{i}\gamma)[\Lambda] + \mathrm{i}[\bar{K_s}]\}\mathbf{Z}(\omega) = \mathbf{F}(\omega) \tag{3}$$

where the mass and stiffness matrices are diagonalized, $[\Phi]^{\mathrm{T}}[M][\Phi] = [I]$ and $[\Phi]^{\mathrm{T}}[K][\Phi] = [\Lambda]$, as a result of the mode orthogonality and mass normalization, and $\mathbf{F}(\omega) = [\Phi]^{\mathrm{T}}\mathbf{P}(\omega) \in \mathbb{C}^{m\times nf}$. However, the modal viscous and structural damping matrices, $[\bar{B}] = [\Phi]^{\mathrm{T}}[B][\Phi]$, $[\bar{K_s}] = [\Phi]^{\mathrm{T}}[K_s][\Phi] \in \mathbb{R}^{m\times m}$ are still fully populated matrices, which results in still expensive cost to solve Eq. (3) due to $O(m^3)$ operations to factor the coefficient matrix at each excitation frequency ω.

3. Modal frequency response analysis for a partially damped structure

For the partially damped structures with viscous and structural damping, which are the problems of interest in this paper, the corresponding finite element matrices $[B]$ and $[K_s]$ result in very sparse. This fact implies that the rank of the viscous and structural damping matrices become very low compared to the mass and stiffness matrices. Therefore, it is essential to identify the rank of these non-proportional damping matrices. Conventionally, the rank of matrix can be obtained with singular value decomposition (SVD) method [8]. An asymmetric matrix should be added to the viscous damping matrix $[B]$ when gyroscopic effects are considered.

First, $[B]$ and $[K_s]$ are condensed to $[B]^c \in \mathbb{R}^{b\times b}$ and $[K_s]^c \in \mathbb{R}^{s\times s}$ that contain only non-zero rows and columns of the finite element matrix $[B]$ and $[K_s]$, respectively. Generally b, which is non-zero rows and columns of $[B]$, and s, which is non-zero rows and columns of $[K_s]$, are much smaller than n since $[B]$ and $[K_s]$ are very sparse for the problem of interest in this paper. Using the condensed matrices, the modal viscous damping matrix $[\bar{B}]$ and the modal structural damping matrix $[\bar{K_s}]$ can be rewritten as

$$[\bar{B}] = [\Phi_b]^{\mathrm{T}}[B]^c[\Phi_b] \tag{4}$$

$$[\bar{K_s}] = [\Phi_s]^{\mathrm{T}}[K_s]^c[\Phi_s] \tag{5}$$

in which $[\Phi_b] \in \mathbb{R}^{b\times m}$ and $[\Phi_s] \in \mathbb{R}^{s\times m}$ contain rows of Φ which correspond to non-zero elements in $[B]$ and $[K_s]$, respectively.

Then, instead of applying the SVD method to the FE matrices $[B]$ and $[K_s] \in \mathbb{R}^{n\times n}$ directly, the SVD method for the condensed viscous and structural damping matrices is performed as

$$[B]^c = [U_b][\Sigma_b][V_b]^{\mathrm{T}} \tag{6}$$

$$[K_s]^c = [U_s][\Sigma_s][V_s]^{\mathrm{T}} \tag{7}$$

in which $[\Sigma_b] \in \mathbb{R}^{r_b\times r_b}$ and $[\Sigma_s] \in \mathbb{R}^{r_s\times r_s}$ are the diagonal matrix of singular values for $[B]^c$ and $[K_s]^c$, respectively, in which r_b and r_s are the rank of $[B]^c$ and $[K_s]^c$. $[U_b]$ and $[V_b] \in \mathbb{R}^{b\times r_b}$ are orthogonal matrices for $[B]^c$, and $[U_s]$ and $[V_s] \in \mathbb{R}^{s\times r_s}$ are orthogonal matrices for $[K_s]^c$.

By substituting Eqs. (6) into (4), and (7) into (5), the modal damping matrices $[\bar{B}]$ and $[\bar{K_s}]$ can be represented as

$$[\bar{B}] = [\Phi_b]^{\mathrm{T}}([U_b][\Sigma_b][V_b]^{\mathrm{T}})[\Phi_b] \tag{8}$$

$$[\bar{K_s}] = [\Phi_s]^{\mathrm{T}}([U_s][\Sigma_s][V_s]^{\mathrm{T}})[\Phi_s] \tag{9}$$

Finally, with Eqs. (8) and (9), the modal frequency response problem (3) is rewritten as

$$\left\{-\omega^2[I] + (1+\mathrm{i}\gamma)[\Lambda] + [\Phi_b^{\mathrm{T}}U_b, \Phi_s^{\mathrm{T}}U_s]\begin{bmatrix} \mathrm{i}\omega\Sigma_b & 0 \\ 0 & \mathrm{i}\Sigma_s \end{bmatrix}\begin{bmatrix} V_b^{\mathrm{T}}\Phi_b \\ V_s^{\mathrm{T}}\Phi_s \end{bmatrix}\right\}\mathbf{Z}(\omega) = \mathbf{F}(\omega) \tag{10}$$

For simplicity, the modal frequency response problem (10) can be represented in the form

$$([D(\omega)] + [U][\Sigma(\omega)][V]^{\mathrm{T}})\mathbf{Z}(\omega) = \mathbf{F}(\omega) \tag{11}$$

where

$$[D(\omega)] = -\omega^2[I] + (1+\mathrm{i}\gamma)[\Lambda] \in \mathbb{C}^{m\times m} \tag{12}$$

$$[U] = [\Phi_b^{\mathrm{T}}U_b, \Phi_s^{\mathrm{T}}U_s] \in \mathbb{R}^{m\times r} \tag{13}$$

$$[V]^{\mathrm{T}} = \begin{bmatrix} V_b^{\mathrm{T}}\Phi_b \\ V_s^{\mathrm{T}}\Phi_s \end{bmatrix} \in \mathbb{R}^{r\times m} \tag{14}$$

$$[\Sigma(\omega)] = \begin{bmatrix} \mathrm{i}\omega\Sigma_b & 0 \\ 0 & \mathrm{i}\Sigma_s \end{bmatrix} \in \mathbb{C}^{r\times r} \tag{15}$$

and $r = r_b + r_s$.

Table 1
The FFRA algorithm for the dynamic response analysis of the partially damped structure in the frequency domain.

Form the frequency independent matrices	
(1a)	form $[B]^c$
(1b)	$[B]^c = [U_b][\Sigma_b][V_b]^T$
(1c)	form $[\Phi_b]$
(2a)	form $[K_s]^c$
(2b)	$[K_s]^c = [U_s][\Sigma_s][V_s]^T$
(2c)	form $[\Phi_s]$
(3)	form $[D], [U], [V]$, and $[\Sigma]$
Frequency loop:	
for $i = 1, 2, \ldots,$ nfreq	
(4a)	$P_1 = [D]^{-1}\mathbf{F}(\omega)$
(5a)	$Q_1 = [V]^T[D]^{-1}\mathbf{F}(\omega)$
(5b)	$Q_2 = ([\Sigma]^{-1} + [V]^T[D]^{-1}[U])^{-1}Q_1$
(5c)	$P_2 = [D]^{-1}[U]Q_2$
(6)	$\mathbf{Z}(\omega) = P_1 - P_2$
end for	

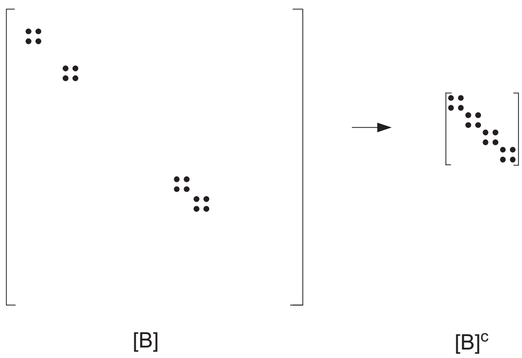

Fig. 2. The sparsity of the finite element viscous damping matrix $[B]$ and the corresponding condensed viscous damping matrix $[B]^c$ (dot: non-zero entry).

The strategy to compute the modal solution $\mathbf{Z}(\omega)$ is to use the Sherman–Morrison–Woodbury (SMW) formula [8] for the inverse of coefficient matrix in Eq. (11), which is subjected to low rank modifications, instead of factoring the coefficient matrix of Eq. (11) with $O(m^3)$ operations at each excitation frequency ω. The general form of the Sherman–Morrison–Woodbury formula [8,9] is

$$(A + X_1 G X_2^T)^{-1} = A^{-1} - A^{-1}X_1(G^{-1} + X_2^T A^{-1} X_1)^{-1} X_2^T A^{-1} \tag{16}$$

Then, the modal solution $\mathbf{Z}(\omega)$ can be expressed in the form

$$\begin{aligned}\mathbf{Z}(\omega) &= ([D] + [U][\Sigma(\omega)][V]^T)^{-1}\mathbf{F}(\omega)\\ &= \{[D]^{-1} - [D]^{-1}[U]([\Sigma]^{-1} + [V]^T[D]^{-1}[U])^{-1}[V]^T[D]^{-1}\}\mathbf{F}(\omega)\end{aligned} \tag{17}$$

Once the modal solution $\mathbf{Z}(\omega)$ is obtained efficiently, because the inversion of $([\Sigma]^{-1} + [V]^T[D]^{-1}[U]) \in \mathbb{C}^{r\times r}$ and other matrix multiplications are inexpensive due to $r \ll m$, the frequency response in the finite element dimension can be obtained from $\mathbf{X}(\omega) = [\Phi]\mathbf{Z}(\omega)$. Table 1 summarizes the details of the FFRA algorithm for the frequency response analysis of partially damped structure.

4. Numerical example and discussion

The newly developed FFRA algorithm for a partially damped structure is validated with an industry passenger automobile finite element model that has 1.3 million degrees of freedom in FE dimension. A commercial FE software NASTRAN [10] is used to model this automobile model. The frequency range of interest is from 1 to 500 Hz with a 1 Hz increment, so that the global cutoff frequency is set to 750 Hz. The numbers of global modes m obtained from the generalized eigenvalue problem $K\Phi = M\Phi\Lambda$ is 3361.

In this FE model, only four viscous damping FE elements and four structural damping FE elements are used to model engine mounts and the suspension springs, respectively. In this FE model, the viscous and structural damping are described with one-dimensional linear FE element, so that the size of each FE damping matrix results in 2×2 [11]. Fig. 2 illustrates the concept of the sparsity of the FE viscous damping matrix $[B]$ of order n, which has only 16 non-zero entries, and the condensed viscous damping matrix $[B]^c$, in which the dimension b becomes 8. Similarly, the FE structural damping matrix $[K_S]$ is reduced to $[K_S]^c$, in which the dimension s results in 8.

Once the condensed matrices, $[B]^c$ and $[K_S]^c$, are obtained, the rank of each matrix is computed. The rank of the condensed viscous and structural damping matrix, r_b and r_s, is identified as 8 and 8, respectively. The computational costs of the singular value decomposition to identify the rank of matrices are very inexpensive, at most $O(r_b^3)$ and $O(r_s^3)$, because the size of the condense matrices is very small compared to the size n of FE matrices, that is, $r_b \ll n$ and $r_s \ll n$.

Based on the singular values and orthogonal matrices obtained from the singular value decomposition of $[B]^c$ and $[K_S]^c$, the modal frequency response problem is reformulated as expressed in Eq. (11). Because this FE model does not include any gyroscopic effect, $[U]$ and $[V]$ are the same. Then, the Sherman–Morrison–Woodbury formula is employed to invert the coefficient matrix. Note that, in Eq. (17), the inversion of diagonal matrix $[D]$ is trivial, and the inversion of $([\Sigma]^{-1} + [U]^T[D]^{-1}[U])$ is also very economical since the dimension, $r = r_b + r_s$, is only 16. Therefore, it is expected that the total cost

Table 2
Elapsed time of the modal frequency response analysis for the FE model.

	FFRA	Direct method (ZSYSV)
elapsed time	2 min 13 sec	1 hr 27 min 52 sec

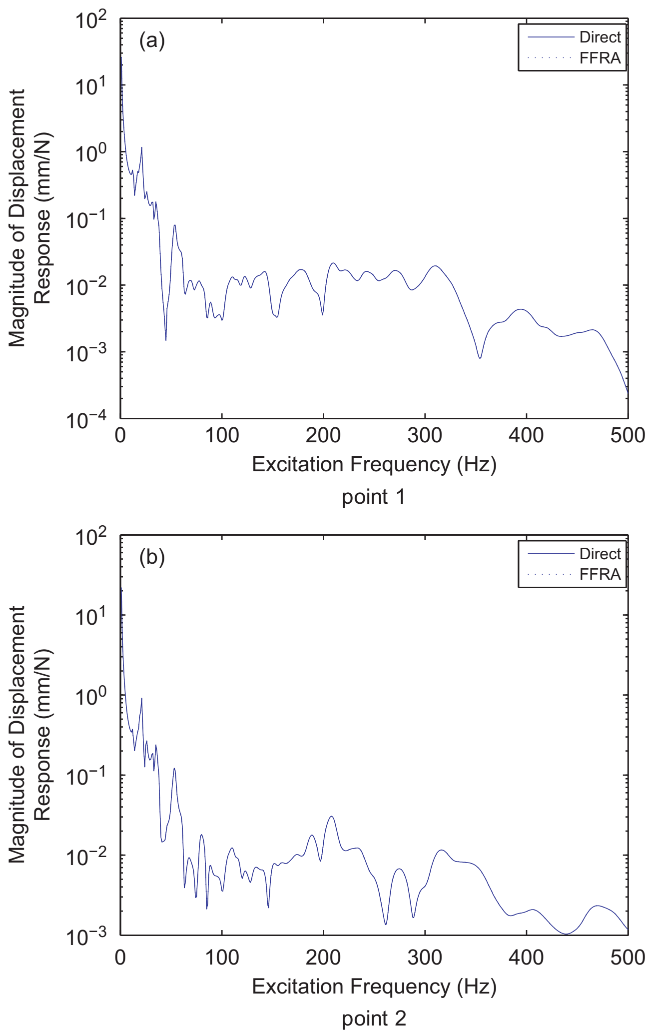

Fig. 3. Dynamic response of displacement in the frequency domain for the FE model. (a) point 1, (b) point 2.

to solve the modal frequency response problem with the new approach can be significantly reduced compared to the traditional approach that factorize the coefficient matrix at each frequency with $O(m^3)$ operations, in which m is 3361.

Table 2 shows the elapsed time of the modal frequency response analysis (3) with the direct method, ZSYSV in LAPACK [12], and the modal frequency response analysis (17) with the FFRA algorithm. An HP rx5670 with 900 MHz Itanium II processor is used to validate the performance and accuracy of the FFRA algorithm. The FFRA algorithm reduced the computational cost, 97.5 percent, compared to the direct method ZSYSV in LAPACK, in which the coefficient matrix is factored at each frequency.

The magnitude of dynamic response in displacement from the FFRA is compared to the results from the direct method ZSYSV in LAPACK in order to validate the accuracy of the FFRA algorithm. Both Figs. 3(a) and (b) show that the FFRA algorithm produce the same results as the direct method. It is obvious outcome because the FFRA algorithm is an efficient problem reformulation [1,5], not an approximation, to obtain a high performance.

5. Conclusions

The fast frequency response analysis algorithm is extended to solve the modal frequency response problem for the structural system with partially distributed structural and viscous damping materials. In this structural system, the finite element viscous and structural damping matrices are typically very sparse, so that the rank of the matrices results in low. Then the modal frequency response problem is reformulated with the low rank matrices of the non-proportional damping, which is obtained from the singular value decomposition method. The reformulated problem is solved with the Sherman–Morrison–Woodbury formula, so the modal response solution is obtained inexpensively without factoring the coefficient matrix at each excitation frequency. Numerical results show that the FFRA algorithm improves the performance significantly, while providing the same accuracy as the direct method.

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